

EXERCISE 1

1 Integrate with respect to x .

a $(1+x)^3$

b $(x+4)^6$

c $(2x+1)^4$

d $(1-x)^4$

e $\frac{1}{(x+3)^2}$

f $2(3x-1)^2$

g $\frac{4}{(x+1)^3}$

h $(5x+3)^{-2}$

i $7e^x$

j $x+5e^x$

k e^{5x}

l $e^{6x}-1$

m $3e^{2x}$

n $9e^{-x}$

o $\frac{4}{e^x}$

p $\frac{1}{e^x}+2x$

2 Integrate with respect to x .

a $\int \sin x \, dx$

b $\int \cos 4x \, dx$

c $\int \sin 10x \, dx$

d $2 \int \cos 5x \, dx$

e $\int \frac{1}{2} \sin 6x \, dx$

f $\int (\cos 2x + 2x) \, dx$

g $\int \sin(4x-1) \, dx$

h $\int [x - \cos(x+1)] \, dx$

i $\int \left(2x + \frac{1}{x}\right) \, dx$

j $\int \left(\cos x + \frac{2}{x}\right) \, dx$

k $\int \frac{1}{4x} \, dx$

l $\int \frac{1}{x+3} \, dx$

m $\int \frac{6}{1+x} \, dx$

n $\int \frac{1}{3x+2} \, dx$

o $\int \left(e^{2x} + \frac{2}{x}\right) \, dx$

3 a Work out $\frac{d}{dx}(4x+1)^{\frac{3}{2}}$.

b Hence find $\int (4x+1)^{\frac{1}{2}} \, dx$.

4 Work out,

a $\frac{d}{dx} \sqrt{5x-2}$

b $\int (5x-2)^{-\frac{1}{2}} \, dx$

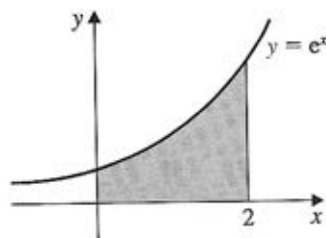
5 Work out the following

a $\int (6x+1)^{\frac{1}{3}} \, dx$

b $\int \frac{1}{\sqrt{x+2}} \, dx$

c $\int (1+2x)^{\frac{3}{2}} \, dx$

6 Find the shaded area under the curve $y = e^x$.



7 Calculate the following indefinite integrals:

a $\int e^{3x+1} \, dx$

b $\int 4e^{2x-3} \, dx$

c $\int \frac{5}{e^x} \, dx$

d $\int (2x+1)^3 dx$

e $\int \frac{1}{(4x-3)^2} dx$

f $\int \frac{e^{2x}+1}{e^x} dx$

g $\int \sqrt{2x+1} dx$

h $\int \frac{1}{\sqrt{4x+5}} dx$

i $\int (e^x+1)^2 dx$

8 Find $\int y dx$ for the following:

a $y = \frac{4}{2x+7}$

b $y = (e^x+1)(e^{-x}+1)$

c $y = \frac{2}{(3x-2)^2}$

d $y = \sqrt{4x+3}$

e $y = \frac{1}{\sqrt{8x-1}}$

f $y = \sqrt[3]{6x+5}$

9 Show that $\frac{x^3+5x+1}{x^2} = x + \frac{5}{x} + \frac{1}{x^2}$.Hence find $\int \frac{x^3+5x+1}{x^2} dx$.

10 Find the following indefinite integrals:

a $\int \left(1 - \frac{2}{x}\right)^2 dx$

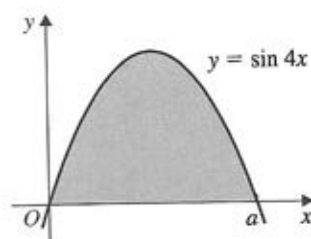
b $\int \frac{x^2+5x+3}{x^2} dx$

c $\int \frac{e^x+1}{2} dx$

d $\int \frac{(x-2)(x-3)}{x^2} dx$

e $\int \frac{(x-5)(x+5)}{x} dx$

f $\int \frac{e^{3x+2}+1}{e^{2x}} dx$

11 The curve $y = \sin 4x$ meets the x -axis at $x=0$ and $x=a$.a Write down the value of a in radians.b Find the area enclosed by the curve $y = \sin 4x$ and the x -axis between $x=0$ and $x=a$.

Evaluate:

12 $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x dx$

13 $\int_0^1 e^{3x} dx$

14 $\int_2^5 \frac{1}{x} dx$

15 $\int_0^{\frac{1}{3}} (3x-1)^3 dx$

16 $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin 2x dx$

17 $\int_{-1}^1 e^{2x+1} dx$

18 $\int_2^4 \frac{10}{x} dx$

19 $\int_1^3 \frac{1}{x+3} dx$

20 $\int_0^5 \sqrt{x+4} dx$

21 $\int_1^{1.5} \cos(2x-1) dx$

22 $\int_1^3 2x + \frac{1}{2x} dx$

23 $\int_0^{\frac{\pi}{4}} \sec^2 x dx$

24 $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x dx$

25 $\int_1^2 \left(x + \frac{1}{x}\right)^2 dx$

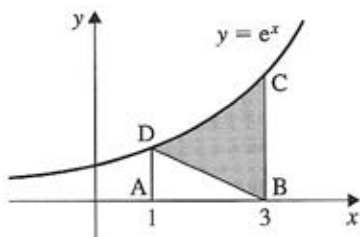
26 $\int_0^2 \frac{1}{2x+1} dx$

- 27 Show that $5 \cos^2 x + 3 \sin^2 x - 1 \equiv \cos 2x + 3$

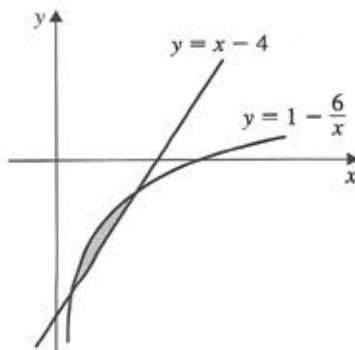
Hence evaluate $\int_0^{\frac{\pi}{4}} (5 \cos^2 x + 3 \sin^2 x - 1) dx$.

- 28 Evaluate $\int_0^{\frac{\pi}{6}} (\cos 3x + \sin 2x) dx$.

- 29 The sketch shows the curve $y = e^x$ and the lines $x = 1$ and $x = 3$.
Show that the area of the shaded region is $e(e^2 - 2)$.



- 30 a Find the x coordinates of the points where the line $y = x - 4$ cuts the curve $y = 1 - \frac{6}{x}$.
b Find the exact area enclosed between the line $y = x - 4$ and the curve $y = 1 - \frac{6}{x}$.



- 31 Find the exact area enclosed between the curve $y = 4 - \frac{3}{x}$ and the line $y = x$.
- 32 a Draw a sketch of the curve $y = \sqrt{9 - x}$ and the line $y = -\frac{1}{5}x + 3$ for x between 0 and 9.
b Show that where the curve and the line meet, $25(9 - x) = (-x + 15)^2$ and solve this to find the x coordinates of the points of intersection.
c Hence find the exact area enclosed between the curve and the line.
- 33* a Show that the area enclosed between $y = e^{2x} + 1$ and $y = 3e^x - 1$ is $\frac{3 - 4 \ln 2}{2}$.
b Show that the area enclosed between $y = 2e^{2x}$ and $y = 18(e^x - 2)$ is $27 - 36 \ln 2$.
- 34* a Find the exact x coordinates (in terms of logarithms) of the points of intersection of the curves $y = (e^x + 1)^2$ and $y = 10e^x - 11$.
b Hence find the exact area enclosed between the two curves.

Example 2

Evaluate the definite integral $\int_1^2 \frac{1}{1-2x} dx$.

$$\begin{aligned}\int_1^2 \frac{1}{1-2x} dx &= -\frac{1}{2} \int_1^2 \frac{-2}{1-2x} dx \\ &= -\frac{1}{2} \left[\ln |1-2x| \right]_1^2 \\ &= -\frac{1}{2} [\ln 3 - \ln 1] \\ &= -\frac{1}{2} \ln 3\end{aligned}$$

We use the modulus function because when $x = 1$ and $x = 2$, $1 - 2x < 0$.

[Remember $|-3| = 3$ and $|-1| = 1$.]

Example 3

Find $\int \tan x dx$.

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{-\sin x}{\cos x} dx \\ &= -\ln \cos x + c \\ &= \ln (\cos x)^{-1} + c \\ &= \ln \sec x + c\end{aligned}$$

[Remember this method.]

EXERCISE 2

Find the following indefinite integrals.

1 $\int \frac{1}{x+3} dx$

2 $\int \frac{2}{2x+1} dx$

3 $\int \frac{2x}{x^2+5} dx$

4 $\int \frac{3x^2}{x^3+2} dx$

5 $\int \frac{1}{4x+1} dx$

6 $\int \frac{1}{7x-1} dx$

7 $\int \frac{e^x}{e^x+3} dx$

8 $\int \frac{x}{x^2+3} dx$

9 $\int \cot x dx$

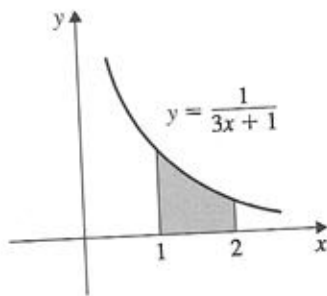
10 $\int \frac{1}{x+1} + \frac{1}{x-2} dx$

11 $\int \frac{2}{2x+1} + \frac{5}{5x+2} dx$

12 $\int \frac{6}{2x+1} - \frac{1}{x+1} dx$

13 Find the area under the curve

$y = \frac{1}{3x-1}$ between $x = 1$ and $x = 2$.



- 14 Find the exact values of the following definite integrals, leaving your answers in terms of logarithms:

a $\int_1^4 \frac{2}{x} dx$

b $\int_1^8 \frac{1}{3x} dx$

c $\int_3^6 \frac{2x+1}{x} dx$

d $\int_2^8 \frac{4x-1}{2x} dx$

e $\int_2^3 \frac{3}{3x-5} dx$

f $\int_2^4 \frac{2}{x-1} dx$

g $\int_1^4 \frac{6}{2x+1} - \frac{3}{3x-2} dx$

h $\int_2^5 \frac{1}{x+1} - \frac{1}{x-1} dx$

i $\int_5^{15} \left(\frac{x+1}{x} \right) dx$

j $\int_0^3 \frac{3}{3x+1} + \frac{2}{2x+3} dx$

k $\int_2^4 \left(\frac{x^2+2x+3}{x} \right) dx$

l $\int_1^3 \left(\frac{x^2-4}{x} \right) dx$

- 15 Obtain the area enclosed by the curve $y = \frac{1}{x}$, the x -axis and the lines $x = -2$ and $x = -3$.

- 16 Evaluate the following, giving your answers correct to two decimal places.

a $\int_{-2}^0 \frac{1}{x-1} dx$

b $\int_1^2 \frac{x}{x^2-9} dx$

c $\int_1^3 \frac{1}{1-4x} dx$

- 17 Find the following

a $\int \tan x dx$

b $\int \frac{\cos 2x}{\sin 2x} dx$

c $\int \frac{\sec^2 x}{\tan x} dx$

d $\int \frac{\sin x}{\cos x+3} dx$

e $\int \frac{2 \sin x \cos x}{\cos 2x} dx$

f $\int \frac{1}{x \ln x} dx$

10.3 Integration by substitution

Some integrals are easier to perform if we change the variable making a substitution.

Let $f(x)$ be a function of x and let $y = \int f(x) dx$

So $\frac{dy}{dx} = f(x)$.

If u is a function of x then y is a function of u .

$$\begin{aligned} \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\ &= f(x) \frac{dx}{du} \end{aligned}$$

Integrating with respect to u , $y = \int f(x) \frac{dx}{du} du$

$$\text{or } \int f(x) dx = \int f(x) \frac{dx}{du} du$$

So an integral with respect to x is transformed into an integral with respect to u .

We substitute for $f(x)$ and $\frac{dx}{du}$ in terms of u .

$$\text{Let } u = 2x - 3 \Rightarrow x = \frac{u+3}{2} \Rightarrow \frac{dx}{du} = \frac{1}{2}$$

When $x = 1$, $u = -1$ and when $x = 2$, $u = 1$

$$\begin{aligned} \therefore \int_1^2 x(2x-3)^3 dx &= \int_{u=-1}^{u=1} x(2x-3)^3 \frac{dx}{du} du \\ &= \int_{-1}^1 \left(\frac{u+3}{2} \right) u^3 \frac{1}{2} du = \frac{1}{4} \int_{-1}^1 u^4 + 3u^3 du \\ &= \frac{1}{4} \left[\frac{u^5}{5} + \frac{3u^4}{4} \right]_{-1}^1 \\ &= \frac{1}{4} \left[\frac{1}{5} + \frac{3}{4} - \left(-\frac{1}{5} + \frac{3}{4} \right) \right] = \frac{1}{4} \left[\frac{2}{5} \right] = \frac{1}{10} \end{aligned}$$

Example 5

Find $\int \frac{1}{\sqrt{1-x^2}} dx$

In this case we use a non linear substitution.

Let $x = \sin u$

$$\frac{dx}{du} = \cos u$$

$$\begin{aligned} \therefore \int \frac{1}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{1-\sin^2 u}} \frac{dx}{du} \cdot du \\ &= \int \frac{1}{\sqrt{\cos^2 u}} \cos u du \\ &= \int 1 du = u + c \\ &= \sin^{-1} x + c \end{aligned}$$

EXERCISE 3

1 Find the following integrals, using the given substitutions.

a $\int x(x+2)^2 dx$ $u = x+2$

b $\int x(x-3)^2 dx$ $u = x-3$

c $\int 2x(x+4)^3 dx$ $u = x+4$

d $\int x^2(x-1)^2 dx$ $u = x-1$

2 Find the following indefinite integrals using the given substitutions:

a $\int x(x+1)^3 dx$ $u = x+1$

b $\int x(x-1)^5 dx$ $u = x-1$

c $\int x\sqrt{x+3} dx$ $u = \sqrt{x+3}$

d $\int \frac{x^2}{\sqrt{x-2}} dx$ $u = x-2$

e $\int x(2x+1)^3 dx$ $u = 2x+1$

f $\int x\sqrt{4x-1} dx$ $u = 4x-1$

g $\int \frac{x}{\sqrt{5x+1}} dx$ $u = 5x+1$

h $\int (x+1)(3x-2)^4 dx$ $u = 3x-2$

3 Find the following indefinite integrals using the given substitutions:

a $\int 12x(x^2 + 1)^3 dx$ $u = x^2 + 1$

b $\int 18x^2(x^3 - 3)^2 dx$ $u = x^3 - 3$

c $\int e^x(e^x - 1)^3 dx$ $u = e^x - 1$

d $\int \frac{e^x}{\sqrt{e^x + 2}} dx$ $u = e^x + 2$

e $\int \sin^2 x \cos x dx$ $u = \sin x$

f $\int \cos^3 x \sin x dx$ $u = \cos x$

g $\int \tan^3 x \sec^2 x dx$ $u = \tan x$

h $\int \frac{1}{\sqrt{1 - x^2}} dx$ $x = \sin \theta$

4 Evaluate the following definite integrals using the given substitutions:

a $\int_0^1 x(x + 1)^3 dx$ $u = x + 1$

b $\int_3^4 x(x - 3)^4 dx$ $u = x - 3$

c $\int_3^6 x\sqrt{x - 2} dx$ $u = x - 2$

d $\int_0^4 \frac{x}{\sqrt{2x + 1}} dx$ $u = 2x + 1$

5 Find the following definite integrals using the given substitutions:

a $\int_0^{\pi} e^{\cos x} \sin x dx$ $u = \cos x$

b $\int_1^3 \frac{x}{\sqrt{x + 2}} dx$ $u = x + 2$

c $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx$ $u = \cos x$

d $\int_0^{\frac{\pi}{2}} \sin x \sin 2x dx$ $u = \sin x$

6 Use the substitution $x = \sin \theta$ to show that, for $|x| \leq 1$,

$$\int \frac{1}{(1 - x^2)^{\frac{3}{2}}} dx = \frac{x}{(1 - x^2)^{\frac{1}{2}}} + c, \text{ where } c \text{ is an arbitrary constant.}$$

7 By making the substitution $x = 2 \sin t$, evaluate $\int_0^1 \frac{x^2}{\sqrt{4 - x^2}} dx$.

Give your answer correct to 2 decimal places. [Hint: see section 10.5]

10.4 Integration by parts

In Part 4 we obtained a formula for differentiating the product of two functions u and v .

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Integrate both sides of the equation with respect to x .

$$\int \frac{d}{dx}(uv) dx = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

$$\text{or } \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

This is the formula for integrating by parts.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = x$

$$\begin{aligned}\therefore I &= (\ln x) \times x - \int x \times \frac{1}{x} dx \\ &\quad \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ u & v & v & \frac{du}{dx} \end{array} \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + c\end{aligned}$$

Example 4

Find $I = \int x^2 e^x dx$

Let $u = x^2$, $\frac{dv}{dx} = e^x$ so that $\frac{du}{dx} = 2x$ and $v = e^x$

$$\therefore I = x^2 e^x - \int e^x \cdot 2x dx$$

Now consider $\int 2x e^x dx$

Let $u = 2x$, $\frac{dv}{dx} = e^x$ so that $\frac{du}{dx} = 2$ and $v = e^x$

$$\begin{aligned}\therefore \int 2x e^x dx &= 2x e^x - \int e^x \cdot 2 dx \\ &= 2x e^x - 2e^x + c\end{aligned}$$

$$\begin{aligned}\therefore I &= x^2 e^x - (2x e^x - 2e^x) + c \\ &= e^x (x^2 - 2x + 2) + c\end{aligned}$$

This example shows that in some questions you may have to integrate by parts more than once.

EXERCISE 4

1 Use integration by parts to find the following.

a $\int x(1+x)^2 dx$ $\left[\text{Let } u = x \text{ and } \frac{dv}{dx} = (1+x)^2 \right]$

b $\int x \cos x dx$ $\left[\text{Let } u = x \text{ and } \frac{dv}{dx} = \cos x \right]$

2 Integrate the following:

a $\int x e^x dx$

b $\int x(1+x)^3 dx$

c $\int x e^{-x} dx$

d $\int x e^{3x} dx$

e $\int x \sin x dx$

f $\int 3x \cos 2x dx$

g $\int 2x(x-1)^3 dx$

h $\int x\sqrt{x+1} dx$

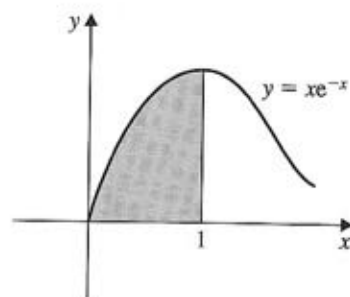
i $\int x \ln 2x dx$

j $\int x^2 \ln x dx$

k $\int \frac{\ln x}{x^3} dx$

l $\int x^2 e^x dx$

- 3 The sketch shows the curve $y = xe^{-x}$ and the line $x = 1$. Find the shaded area.



- 4 Calculate the exact values of following definite integrals.

a $\int_0^1 xe^{-3x} dx$

b $\int_0^{\frac{\pi}{4}} x \sin 2x dx$

c $\int_0^1 x(2x+1)^2 dx$

d $\int_3^8 x\sqrt{x+1} dx$

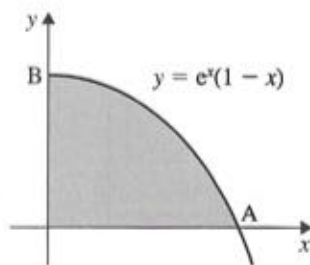
e $\int_1^e x^2 \ln x dx$

f $\int_0^{\frac{\pi}{2}} x^2 \sin x dx$

- 5 a Find the exact value of $\int_0^1 x(2x+1)^3 dx$ by using integration by parts.
 b Show that you get the same value for $\int_0^1 x(2x+1)^3 dx$ by using the method of integration by substitution, with $u = 2x+1$.
- 6 a Show that $\int 3x(1+2x)^4 dx = \frac{(1+2x)^5(10x-1)}{40} + c$.
 b Show that $\int \frac{x}{e^{2x}} dx = -\left(\frac{2x+1}{4e^{2x}}\right) + c$.
 c Show that $\int \frac{4x}{\sqrt{x+1}} dx = \frac{8(x-2)\sqrt{x+1}}{3} + c$.

- 7 The curve $y = e^x(1-x)$ is shown.

- a Write down the coordinates of point A and point B where the curve cuts the axes.
 b Calculate the area enclosed by the curve and the x and y axes.



- 8 a Find $\frac{d}{dx}(e^{x^2})$.
 b By writing $x^3 e^{x^2} = \frac{x^2}{2}(2x e^{x^2})$, find $\int x^3 e^{x^2} dx$.

9* a Show that $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$.

b Find a similar expression for $\int e^x \sin x \, dx$.

c Hence find $\int e^x \cos x \, dx$.

10* Find, a $\int e^{5x} \sin 3x \, dx$.

b $\int e^{ax} \sin bx \, dx$.

10.5 Using trigonometric identities

Sometimes we use trigonometric identities to transform an integral into a function which can be integrated. The most frequently used formulae are below:

$$\cos 2x = 2 \cos^2 x - 1$$

$$\cos 2x = 1 - 2 \sin^2 x$$

$$1 + \tan^2 x = \sec^2 x$$

Remember also that $\cos 4x = 2 \cos^2 2x - 1$ and so on.

Example 1

Find $I = \int \sin^2 x \, dx$

Rearrange the formula $\cos 2x = 1 - 2 \sin^2 x$
 $2 \sin^2 x = 1 - \cos 2x$
 $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\begin{aligned}\therefore I &= \frac{1}{2} \int (1 - \cos 2x) \, dx = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + c \\ &= \frac{1}{2}x - \frac{1}{4} \sin 2x + c\end{aligned}$$

Example 2

Find $I = \int \cos^2 4x \, dx$

Rearrange the formula $\cos 8x = 2 \cos^2 4x - 1$
 $\cos^2 4x = \frac{1}{2}(\cos 8x + 1)$

$$\therefore I = \frac{1}{2} \int (\cos 8x + 1) \, dx = \frac{1}{16} \sin 8x + \frac{1}{2}x + c$$

Example 3

Find $I = \int \tan^2 x \, dx$

From above $\tan^2 x = \sec^2 x - 1$

$$\begin{aligned}\therefore I &= \int (\sec^2 x - 1) \, dx \\ &= \tan x - x + c\end{aligned}$$

EXERCISE 5

1 Integrate with respect to x :

a $\int \sin^2 x \, dx$

b $\int \cos^2 x \, dx$

c $\int \tan^2 x \, dx$

d $\int \sin^2 3x \, dx$

e $\int \cos^2 2x \, dx$

f $\int (1 - 2 \sin x)^2 \, dx$

g $\int (\cos x + \sec x)^2 \, dx$

h $\int \tan^2 2x \, dx$

i $\int (1 + \tan x)^2 \, dx$

2 Evaluate the following

a $\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$

b $\int_0^{\frac{\pi}{2}} \cos^2 \frac{1}{2} x \, dx$

c $\int_0^{\frac{\pi}{3}} \tan^2 x \, dx$

d $\int_0^{\frac{\pi}{4}} (1 + 2 \cos 2x)^2 \, dx$

e $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x) \, dx$

$$\begin{aligned}
 \therefore I &= \int_2^3 \frac{4}{2x-1} + \frac{3}{(2x-1)^2} + \frac{1}{x-1} dx \\
 &= \left[2 \ln |2x-1| - \frac{3}{2}(2x-1)^{-1} + \ln |x-1| \right]_2^3 \\
 &= 2 \ln 5 - \frac{3}{2} \times \frac{1}{5} + \ln 2 - \left(2 \ln 3 - \frac{3}{2} \times \frac{1}{3} + \ln 1 \right) \\
 &= \ln 5^2 + \ln 2 - \ln 3^2 - \frac{3}{10} + \frac{1}{2} \\
 &= \ln \left(\frac{50}{9} \right) + \frac{1}{5}
 \end{aligned}$$

EXERCISE 7

1 a Express $\frac{x}{(1+x)(1+2x)}$ in partial fractions.

b Find $\int \frac{x}{(1+x)(1+2x)} dx$.

2 Using partial fractions, calculate the following indefinite integrals:

a $\int \frac{2}{x^2-1} dx$

b $\int \frac{x+6}{x^2+9x+20} dx$

c $\int \frac{x+18}{x^2-9x+14} dx$

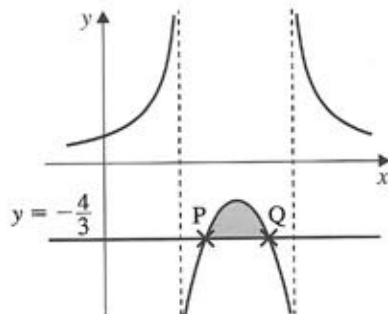
3 Use partial fractions to show that $\int_0^1 \frac{18-4x-x^2}{(4-3x)(1+x)^2} dx = \frac{7}{3} \ln 2 + \frac{3}{2}$.

4 The diagram shows the graph of

$$y = \frac{1}{(x-1)(x-3)}$$

The line $y = -\frac{4}{3}$ intersects the curve at points P and Q.

Calculate the area of the shaded region, giving your answer correct to three decimal places.



5 a Express $\frac{8x^2+16x+7}{(x+1)^2(2x+1)}$ in partial fractions.

b Hence show that $\int_0^1 \frac{8x^2+16x+7}{(x+1)^2(2x+1)} dx = \frac{1}{2} + 2 \ln 6$.

6 a Express $\frac{6x^3-7x^2+14x-7}{(2x-1)(3x-2)}$ in partial fractions.

b Hence show that $\int_1^2 \frac{6x^3-7x^2+14x-7}{(2x-1)(3x-2)} dx = \frac{3}{2} + \ln 12$.

- 7 Show that $\int_2^3 \frac{2x^3 - 9x^2 + 11x - 9}{(2x - 7)(x - 1)} dx = \frac{5}{2} + \ln\left(\frac{2}{3}\right)$.
- 8 Show that $\int_2^4 \frac{4x^4 - 28x^3 + 51x^2 - 39x + 1}{(2x - 1)^2(x - 5)} dx = \frac{88}{21} - \frac{3}{2} \ln 3 + \frac{1}{2} \ln 7$
- 9 $f(x) = x^3 + 6x^2 + 11x + 6$
 A root of the equation $f(x) = 0$ is -2 .
 a Using algebra and showing all your working, factorise $f(x)$ completely.
 b Express $\frac{2}{f(x)}$ in partial fractions.
 c Evaluate $\int_0^1 \frac{2}{f(x)} dx$, giving your answer in the form $\ln\left(\frac{a}{b}\right)$ where a and b are integers.
- 10 Using the substitution $u = e^x$ and partial fractions, show that

$$\int_0^{\ln 2} \frac{1}{1 + e^x} dx = \ln\left(\frac{4}{3}\right)$$

10.8 Integration methods, summary

One of the main problems in integration is in deciding which of several different methods to use. In this section we will briefly review six main types of integral and provide a large number of mixed questions so that you can practise making decisions about the best method to adopt. Use the exercise sensibly, it is not necessary to do *all* the questions.

Type 1 Standard functions

$$\begin{aligned} \int \cos ax \, dx &= \frac{1}{a} \sin ax, & \int \sin ax \, dx &= -\frac{1}{a} \cos ax, \\ \int \sec^2 x \, dx &= \tan x, & \int e^{ax} \, dx &= \frac{1}{a} e^{ax}, & \int \frac{1}{x} \, dx &= \ln x \\ \int (ax + b)^n \, dx &= \frac{1}{a(n+1)} (ax + b)^{n+1} \end{aligned}$$

Examples

a $\int \sin 3x \, dx = -\frac{1}{3} \cos 3x + c$

b $\int \sec^2 5x = \frac{1}{5} \tan 5x + c$

c $\int e^{7x-1} \, dx = \frac{1}{7} e^{7x-1} + c$

d $\int \frac{1}{3x+1} \, dx = \frac{1}{3} \ln(3x+1) + c$

e $\int (2x-3)^4 \, dx = \frac{1}{10} (2x-3)^5 + c$

f $\int (5x+3)^{\frac{1}{2}} = \frac{2}{15} (5x+3)^{\frac{3}{2}} + c$

Type 2 The form $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$

Examples

a $\int \frac{2x}{x^2+7} dx = \ln(x^2+7) + c$

b $\int \frac{x^2}{x^3-5} = \frac{1}{3} \int \frac{3x^2}{x^3-5} = \frac{1}{3} \ln(x^3-5) + c$

c $\int \frac{\cos x}{\sin x} dx = \ln \sin x + c$

d $\int \frac{e^x}{e^x+11} dx = \ln(e^x+11) + c$

Type 3 Using trigonometric identities

For $\int \sin^2 x dx$ or $\int \cos^2 x dx$ use $\cos 2x = 2 \cos^2 x - 1$
or $\cos 2x = 1 - 2 \sin^2 x$

For $\int \tan^2 x dx$ use $1 + \tan^2 x = \sec^2 x$

Examples

a $\int \cos^2 x dx = \frac{1}{2} \int (\cos 2x + 1) dx = \frac{1}{2} \left(\frac{1}{2} \sin 2x + x \right) + c$

b $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + c$

Type 4 By substitution

Example

$\int x(1+x)^3 dx$

Let $u = 1 + x$

$\therefore x = u - 1$

$\frac{dx}{du} = 1$

$$\begin{aligned} \int x(1+x)^3 \frac{dx}{du} du &= \int (u-1)u^3 du = \frac{u^5}{5} - \frac{u^4}{4} + c \\ &= \frac{4u^5 - 5u^4}{20} + c \\ &= \frac{(1+x)^4}{20} [4(1+x) - 5] + c \\ &= \frac{(1+x)^4}{20} (4x - 1) + c \end{aligned}$$

Type 5 Integration by parts

$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Example

$$\int x e^x dx \quad \text{Let } u = x \quad \text{and} \quad \frac{dv}{dx} = e^x$$

$$\text{Then } \frac{du}{dx} = 1 \quad \text{and} \quad v = e^x$$

$$\begin{aligned} \therefore \int x e^x dx &= x e^x - \int e^x \times 1 dx \\ &\quad \begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \uparrow \\ u & \frac{dv}{dx} & & v & \frac{du}{dx} \end{array} \\ &= x e^x - e^x + c \end{aligned}$$

Type 6 Using partial fractions

Example

$$\begin{aligned} \int \frac{2+3x}{(1+x)(1+2x)} dx &= \int \frac{1}{1+x} + \frac{1}{1+2x} dx && \text{(working omitted)} \\ &= \ln(1+x) + \frac{1}{2} \ln(1+2x) + c \end{aligned}$$

EXERCISE 8

Section A Find the following.

1 $\int (x+2)^3 dx$

2 $\int \cos 3x dx$

3 $\int 4e^x dx$

4 $\int \frac{1}{x} dx$

5 $\int \frac{1}{x^2} dx$

6 $\int \frac{2}{x^2-1} dx$

7 $\int x \cos x dx$

8 $\int \frac{5}{5x-1} dx$

9 $\int \sec^2 2x dx$

10 $\int \frac{2x}{x^2+a} dx$

11 $\int x(x+4)^2 dx$

12 $\int x e^{5x} dx$

13 $\int \frac{\cos 3x}{\sin 3x} dx$

14 $\int e^{3x+2} dx$

15 $\int x^3 \sqrt{x^4-1} dx$

16 $\int \frac{5}{x-7} dx$

17 $\int \frac{5x+7}{(x^2+3x+2)} dx$

18 $\int \cos^2 x dx$

19 $\int x \ln x dx$

20 $\int \frac{x+1}{x} dx$

21 $\int \frac{1}{1+x} dx$

22 $\int x^2 e^x dx$

23 $\int \sin(3-4x) dx$

24 $\int \frac{4}{\cos^2 x} dx$

25 $\int (e^x - e^{-x})^2 dx$

26 $\int \frac{x^2+3x+1}{x} dx$

27 $\int \tan x dx$

28 $\int \tan^2 x dx$

29 $\int x(x^2+1)^3 dx$

30 $\int x \sin 2x dx$

31 Integrate by using the given substitution.

a $\int 3x^2 e^{x^3} dx$ $u = x^3$

b $\int \cos x e^{\sin x} dx$ $u = \sin x$

c Show that $\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{8} + \frac{1}{4}$ [Let $x = \tan u$]

d Evaluate $\int_2^{10} \frac{x}{\sqrt{2x+5}} dx$ $u^2 = 2x+5$

32 Find the exact values of the following:

a $\int_0^1 e^x dx$

b $\int_0^2 \frac{e^x}{2} dx$

c $\int_1^3 \frac{1}{x} dx$

d $\int_1^4 \frac{1}{2x} dx$

33 Find the exact values of the following:

a $\int_1^3 \frac{x+1}{x} dx$

b $\int_1^2 \left(\frac{x+2}{x} \right)^2 dx$

c $\int_3^6 \left(\frac{1}{x} + 3 \right)^2 dx$

d $\int_1^{e^2} \left(\frac{e}{x} + 1 \right)^2 dx$

34 Given that $\frac{dy}{dx} = \frac{1}{2x}$ and that $y = 3$ when $x = 1$, find y in terms of x .

35 Given that $\frac{dy}{dx} = \frac{5}{3x}$ and that $y = 2$ when $x = e$, find y in terms of x .

36 Given that $\frac{dy}{dx} = \frac{x^2+1}{3x}$ and that $y = 1$ when $x = 1$, find y in terms of x .

37 Given that $\frac{dy}{dx} = e^x + 1$ and that $y = e$ when $x = 1$, find y in terms of x .

38 a Express $\cos 2x$ in terms of $\sin x$.

b Hence find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x dx$.

39 a Show that $\tan^2 x + 1 = \sec^2 x$.

b Use this to find $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2 x dx$.

40 Find $\int x^2 \sin x dx$ by integrating by parts twice.

Section B

1 Calculate the exact values of following definite integrals:

a $\int_0^{\pi} \sin x \, dx$

b $\int_0^{\frac{\pi}{2}} \cos 2x \, dx$

c $\int_0^{\frac{\pi}{12}} \sin 3x \, dx$

d $\int_0^{\frac{\pi}{3}} \sec^2 x \, dx$

e $\int_0^1 e^{2x+1} \, dx$

f $\int_0^{\infty} e^{-3x} \, dx$

g $\int_1^2 (5x-1)^2 \, dx$

h $\int_4^{12} \sqrt{2x+1} \, dx$

i $\int_2^{10} \frac{1}{\sqrt{x-1}} \, dx$

Find the following indefinite integrals.

2 $\int \frac{3}{2x+5} \, dx$

3 $\int 2x e^{x^2} \, dx$

4 $\int \frac{\cos x}{4 + \sin x} \, dx$

5 $\int \frac{3x-6}{x(x-3)} \, dx$

6 $\int 3x \ln x \, dx$

7 $\int x(1+x)^{10} \, dx$

8 $\int \ln x \, dx$

9 $\int 3e^{-3x} \, dx$

10 $\int \sin\left(\frac{x}{2}\right) \, dx$

11 $\int \frac{x}{x+3} \, dx$

12 $\int \cos x - \tan x \, dx$

13 $\int \sin^2 2x \, dx$

14 $\int \sqrt{5x+1} \, dx$

15 $\int \frac{1}{\sqrt{x-4}} \, dx$

16 $\int 3x(2x-1) \, dx$

Find the exact values of the following definite integrals:

17 $\int_0^1 e^x + e^{-x} \, dx$

18 $\int_1^2 (e^x + 1)^2 \, dx$

19 $\int_1^3 \frac{(e^x + 1)}{e^x} \, dx$

20 $\int_1^3 (3x-2)^2 \, dx$

21 $\int_0^1 (5x+1)^3 \, dx$

22 $\int_{-2}^3 (4x-3)^4 \, dx$

23 $\int_1^2 \frac{4}{(2x-1)^3} \, dx$

24 $\int_0^{\frac{1}{3}} \frac{4}{(3x+2)^2} \, dx$

25 $\int_{0.8}^2 \frac{4}{(5x-2)} \, dx$

26 $\int_0^1 x e^{3x} \, dx$

27 $\int_0^{\pi} x \cos x \, dx$

28 $\int_1^2 x \ln x \, dx$

29 $\int_5^8 x\sqrt{9-x} \, dx$

30 $\int_1^4 \frac{2}{x} \, dx$

31 $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \, dx$

32 a If $u^2 = x + 1$ then show that $2u \frac{du}{dx} = 1$.

b Use the substitution $u^2 = x + 1$ to find $\int_0^3 \frac{x}{\sqrt{x+1}} \, dx$.

c Show that you get the same value for $\int_0^3 \frac{x}{\sqrt{x+1}} \, dx$ when you use the substitution $u = x + 1$.

33 Use the substitution $u = \ln x$ to find $\int_1^e \frac{\ln x}{x} \, dx$.

34 a Show that $1 + \cot^2 x = \operatorname{cosec}^2 x$.

b Use this to find $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot^2 x \, dx$.

35 a Calculate $\frac{d}{dx} \left((1 - x^2)^{\frac{3}{2}} \right)$ and so find $\int x\sqrt{1 - x^2} \, dx$.

b Calculate $\int x^3\sqrt{1 - x^2} \, dx$ by considering $x^3\sqrt{1 - x^2} = x^2(x\sqrt{1 - x^2})$.

36 Show that $\int_1^5 \frac{6x + 11}{(2x + 3)(x + 2)} \, dx = \ln\left(\frac{1183}{75}\right)$.

37 Use the substitution $x = \sin u$ to show that $\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + c$

38 Evaluate $\int_0^{\ln 4} \left(\frac{e^{2x} + e^x}{e^x} - 1 \right) \, dx$

39 Find $\int \sec 2x \, dx$

10.9 Differential equations

A first order differential equation contains $\frac{dy}{dx}$.

Consider the differential equation $\frac{dy}{dx} = 2x + 1$.

Integrating both sides with respect to x ,

$$y = x^2 + x + c, \text{ where } c \text{ is a constant.}$$

This is the *general solution* of the differential equation.

If we are given that $y = 10$, when $x = 2$

$$\text{then } 10 = 2^2 + 2 + c$$

$$c = 4$$

$$\text{and } y = x^2 + x + 4$$

This is the *particular solution* of the differential equation.

EXERCISE 9

1 Find the general solution of each equation.

a $\frac{dy}{dx} = \frac{x + 1}{x}$

b $\frac{dy}{dx} = 3x^2 - 1$

c $\frac{dy}{dx} = e^x - 1$

d $\frac{dy}{dx} = \cos x$

e $\frac{dy}{dx} = \frac{e^x + 3}{e^x}$

f $\frac{dy}{dx} = \frac{\cos x}{\sin x}$

EXERCISE 10

1 Find the general solutions of the following equations.

a $\frac{dy}{dx} = x^2 y$

b $\frac{dy}{dx} = \frac{x}{y}$

c $\frac{dy}{dx} = \frac{y}{x}$

d $x \frac{dy}{dx} = x + 1$

e $2y \frac{dy}{dx} = 3x^2$

f $\frac{dy}{dx} = \frac{e^{4x}}{y}$

g $\frac{dy}{dx} = e^{xy^2}$

h $\frac{dy}{dx} = \frac{\cos x}{y}$

i $\frac{dy}{dx} = \frac{1}{1+x}$

j $\frac{dy}{dx} = 2y$

k $\frac{dy}{dx} = 3x^2(3+y)$

l $x \frac{dy}{dx} = \sec y$

m $\frac{dy}{dx} = 4x e^{-y}$

n $\frac{dy}{dx} - 2xy = 0$

o $\frac{dy}{dx} = y^2$

2 Solve the equation $x \frac{dy}{dx} = 2y$, given that $y = 4$ when $x = 1$.

Express y in terms of x .

3 Solve the following differential equations, given the initial conditions:

a $\frac{dx}{dt} = 3x$, given $x = 10$, when $t = 0$

b $\frac{dy}{dt} = -10y$, given $y = 1000$, when $t = 0$

c $\frac{dx}{dt} = x - 5$, given $x = 20$ when $t = 0$

d $-\frac{dy}{dt} = y + 8$, given $y = 10$ when $t = 0$.

4 Solve the following differential equations, giving x or y as a function of t .

a $\frac{dx}{dt} = 5x$, where $x = 100$, when $t = 0$

b $\frac{dy}{dt} = -2y$, where $y = 1000$, when $t = 0$

c $\frac{dx}{dt} = 3x$, where $x = x_0$, when $t = 0$

d $\frac{dy}{dt} = -4y$, where $y = y_0$, when $t = 0$

e $\frac{dx}{dt} = kx$, where $x = x_0$, when $t = 0$

f $-\frac{dy}{dt} = ky$, where $y = y_0$, when $t = 0$.

5 Find the solution of following differential equations.

a $\frac{dy}{dx} = x e^y$, given that $y = 0$ when $x = 2$

b $\frac{dy}{dx} = y \sin x$, given that $y = 1$ when $x = \frac{\pi}{2}$

c $\frac{dy}{dx} = 2xy + 5x$, given that $y = -1$ when $x = 0$

d $y \frac{dy}{dx} = \tan x$, given that $y = 1$ when $x = \frac{\pi}{4}$

e $\frac{dy}{dx} = \sin x \cos^2 y$, given that $y = \frac{\pi}{4}$ when $x = 0$

f $\frac{dy}{dx} = xy + x$, given that $y = 1$ when $x = 1$

6 a Solve the differential equation

$$\frac{dx}{dt} = \frac{20 - x}{10}$$

given that $x = 5$ at $t = 0$.

b Find the value of t for which $x = 10$, giving your answer to three significant figures.

7 Solve the differential equation $\frac{dy}{dx} = y^2 e^{-2x}$, given that $y = 1$ when $x = 0$.

Give your answer in a form expressing y in terms of x .

8 a Express $\frac{1}{(1+x)(2+x)}$ in partial fractions.

b Hence find the solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{(1+x)(2+x)}, \quad x > -1,$$

given that $y = 1$ when $x = 2$.

Express your answer in the form $y = f(x)$.

9 Given x and y are positive, $\frac{dy}{dx} + \frac{y}{x^2} = 0$ and $y = e$ when $x = 1$, show that the solution may be written in the form $y = e^{\frac{1}{x}}$.

10 Show that the general solution of the differential equation

$$(1+x)(1+2x) \frac{dy}{dx} = x \tan y \text{ may be written in the form } \sin y = k \left(\frac{1+x}{\sqrt{1+2x}} \right).$$